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Quantum field theory  
with and without conical singularities:  
Black holes with cosmological constant  
and the multihorizon scenario

Feng-Li Lin<sup>a</sup>

Department of Physics,  
University of Utah,  
Salt Lake City, UT84112-0830, U.S.A.

and

Chopin Soo<sup>b</sup>

National Center for Theoretical Sciences,  
P. O. Box 2-131, Hsinchu, Taiwan 300, Taiwan.

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Boundary conditions and the corresponding states of a quantum field theory depend on how the horizons are taken into account. There is ambiguity as to which method is appropriate because different ways of incorporating the horizons lead to different results. We propose that a natural way of including the horizons is to first consider the Kruskal extension and then define the quantum field theory on the Euclidean section. Boundary conditions emerge naturally as consistency conditions of the Kruskal extension. We carry out the proposal for the explicit case of the Schwarzschild-de Sitter manifold with two horizons. The required period  $\beta$  is the interesting condition that it is the lowest common multiple of  $2\pi$  divided by the surface gravity of both horizons. Restricting the ratio of the surface gravity of the horizons to rational numbers yields finite  $\beta$ . The example also highlights some of the difficulties of the off-shell approach with conical singularities in the multihorizon scenario; and serves to illustrate the much richer interplay that can occur among horizons, quantum field theory and topology when the cosmological constant is not neglected in black hole processes.

Electronic addresses:<sup>a</sup> linfl@mail.physics.utah.edu; <sup>b</sup> cpsoo@phys.nthu.edu.tw

## I. Introduction.

The problem of how quantum field theory with Schwarzschild-de Sitter (S-dS) base manifold[1] is defined is interesting from many different angles. Recent high redshift Type Ia supernovae observations strongly support the presence of a positive cosmological constant [2]. In black hole processes, it is physically relevant to take into account the effects of the cosmological constant,  $\lambda$ . It inevitably arises as the coefficient of a counterterm for quantized matter fields in background spacetimes. There are various indications that the inclusion of the cosmological constant may affect even the qualitative features of black hole processes. Topologically, the Euclidean S-dS manifold with two conical singularities has Euler number  $\chi = 4$  and is not deformable to the pure Schwarzschild solution, which has  $\chi = 2$ , by tuning the cosmological constant. Naive thermodynamic arguments suggest that the pure black hole configuration cannot be obtained as the smooth thermodynamic limit  $\lambda \rightarrow 0$  of the S-dS configuration with two horizons since the size of the outer cosmological horizon becomes infinitely large then and should contribute infinite entropy as the cosmological constant goes to zero, whereas the pure black hole lacks the outer horizon altogether.

In recent years, methods have been developed to regularize the contributions of conical manifolds [3, 4]. They allow, for instance, the discussion of the thermodynamics of black holes in the off-shell approach in which the Hawking temperature for the Schwarzschild black hole is derived from the thermal equilibrium condition given by the extremum of the Euclidean Einstein-Hilbert action [5]. The off-shell approach also makes it feasible to decouple the inverse of the temperature,  $\beta$ , from the Hamiltonian which depends on the mass of the black hole by lifting the on-shell restriction  $\beta = 8\pi m$ . If conical singularities are allowed, we can consider more complicated scenarios to test if the formalism leads to difficulties which are not encountered in the case of the pure black hole. S-dS with two horizons appears to be a particularly relevant example.

In the region between but not including the horizons of S-dS, a single global coordinate patch exists. *The question is how one takes into account the horizons.* In the method with conical singularities, the horizons of S-dS are to be included as conical singularities. However, an important difference for this multihorizon situation is that there is no straightforward way to define the usual on-shell thermal equilibrium temperature as in the case of the pure black hole[6] because it is impossible to simultaneously eliminate the conical defects on both horizons by a single choice of periodicity within the formalism with conical singularities. From this perspective, the strategy of allowing for conical singularities therefore seems rather pertinent and also needed for a multihorizon scenario such as the S-dS since it permits adopting a *single* off-shell periodicity which does not need to coincide with either of the values required to remove conical defects at the horizons. It also seems to imply that an off-shell discussion of thermodynamics is possible despite the apparent unequal intrinsic periodicities of the horizons.

However, as we shall see, this is not the only way to resolve the impasse. Actually, the method does not seem to give the correct results even for the pure de Sitter case. There are also questions with regard to the consistency, or at least ambiguity, of quantum field theories defined on manifolds with conical singularities. In the one-loop effective action in background spacetimes, there are terms involving the square of the curvatures which are divergent and cannot be removed in the off-shell approach with conical singularities precisely because in this formulation, no single choice of  $\beta$  can simultaneously get rid of both conical defects at the horizons. For the pure black hole, one can take the on-shell limit after the computations are done to eliminate the unwanted terms[5].

In Ref.[6], it is suggested that we can consider partitioning the volume into two regions which are in equilibrium with the respective inner and outer horizons. We are then able to do thermodynamics without conical singularities. However, the partition is by no means natural. Moreover, it is very much unlike a patching condition in that the physics depends on how the partition is chosen. Half of the total volume at each natural temperature of the horizons is clearly different from one-third of the volume at one temperature and two-thirds of it at the other. On the other hand, we may even argue that the physical situation of S-dS may correspond more closely to a situation with temperature gradient and even non-equilibrium physics since the natural surface temperatures of the horizons are different.

It is interesting to note that either extremes can have dramatic implications for black hole processes. If conical singularities are allowed, they may be potentially significant, both as remnants of black hole evaporation and seeds for black hole condensation in a de Sitter universe with conical singularities, and can actually serve to preserve the information of the topological Euler number during these processes. It may be possible for a black hole of the S-dS type to achieve the zero mass limit with two conical singularities and a remaining outer horizon, and still maintain the  $\chi = 4$  condition. Moreover the remaining outer horizon could be larger than the sum of the initial black hole and cosmological horizons. This could be consistent with information loss without violating topological conservation laws. On the other hand, if conical singularities are to be excluded, then the mere introduction of the cosmological constant, which can also be induced from quantized matter, could lead to non-equilibrium processes with deviations from blackbody spectrum and its implications for the information loss paradox, due to the presence of two horizons with unequal surface gravity. But neither of these simple extreme scenarios may be entirely correct. The issue of how to define, say quantum field theory, on such a background with multiple horizons has yet to be settled.

The imposed boundary conditions and corresponding states of the quantum field theory depend on how the horizons are accounted for. So it is pertinent to ask if there are *natural* ways to incorporate the horizons. In this paper, we

compare the scenarios with and without conical singularities and illustrate some of the difficulties that are present in the former. We return to the Kruskal extension of the pure black hole solution and observe that there is a generalization for S-dS which will *naturally* incorporate the horizons. The Euclidean quantum field theory is then defined without conical singularities but with patching and consistency conditions which determine the feasible states. When applied to the pure black hole and  $S^4$  de Sitter configurations, the proposal yields the correct Gibbons-Hawking temperatures.

## II. Conical singularities and QFT in S-dS spacetime.

In finite temperature quantum field theories in flat spacetime, temperature dependence of the effective action is introduced through radiative loop corrections and resummation [7]. However, in curved spacetimes there is temperature dependence in the action even at tree level through the periodicity of the Euclideanized metric.

In the formulation with conical singularities, *the horizons are accounted for as conical singularities*[5]. The Euclidean Einstein-Hilbert action which includes contributions from conical singularities is

$$I_g = -\frac{1}{16\pi} \int_{M/\Sigma} d^4x \sqrt{g}(R - 2\lambda) - \frac{1}{16\pi} \int_{\Sigma} d^4x \sqrt{g}R - \frac{1}{8\pi} \int_{\partial M} d^3x \sqrt{h}(K - K_0). \quad (1)$$

Here,  $\Sigma$  denotes the singular set of the horizons due to conical defects, and  $K$  is the second fundamental form.

The partition function and effective action are defined through the path-integral with

$$e^{-I_{eff}(\beta)} = Z(\beta) = \int [\mathcal{D}g][\mathcal{D}\phi] e^{-I_g - I_m}. \quad (2)$$

It is assumed that the period of the Euclidean time variable (which does not need to coincide with either of the values required to remove conical defects at the horizons) is  $\beta \equiv 1/T$ .  $I_m$  is the matter action for  $\phi$  while  $I_g$  is the gravitational action. Thermodynamic information may be extracted from the partition function. For example, Fursaev et al [5] derived the Hawking temperature and Bekenstein-Hawking entropy for the pure Schwarzschild black hole through this formulation from the extremum of the effective action.

We are interested in the case of a black hole with positive cosmological constant i.e. the S-dS configuration, and the Euclidean region between the two horizons. There are two horizons with the larger cosmological horizon at  $r_+$  and the inner black hole horizon at  $r_-$  if we impose the restriction  $9m^2\lambda < 1$  [1, 6]. The region between the inner black hole and outer cosmological horizons also serves as a natural volume for thermodynamic considerations. For the Euclidean S-dS configuration, there are no boundary terms in Eq.(1). This is in contradistinction with the pure Schwarzschild case where the boundary term at infinity contributes

to the Arnowitt-Deser-Misner mass of the black hole. The Euclidean section of interest has a conical singularity at each horizon and their contributions to the Einstein-Hilbert action are taken into account by the second term in Eq.(1).

Specifically, the Euclidean S-dS metric is

$$ds_E^2 = h(r)d\tau^2 + \frac{dr^2}{h(r)} + r^2d\Omega^2, \quad (3)$$

$$h(r) = 1 - \frac{2m}{r} - \frac{\lambda r^2}{3}. \quad (4)$$

Here  $\tau$  is the periodic coordinate with periodicity equal to  $\beta$ . As stated earlier, there are conical singularities on the horizons when  $T$  is not equal to the individual Gibbons-Hawking temperatures associated with the horizons. To reveal the conical singularities on the horizons, we may choose the local coordinate patches and change variables through

$$h(r) = k^2 X^2. \quad (5)$$

The metric becomes

$$ds_E^2 = X^2 d(k\tau)^2 + \frac{dX^2}{(\frac{h'}{2k})^2} + r^2d\Omega^2. \quad (6)$$

$V'$  denotes the first derivative of  $V(r)$  with respect to  $r$ . Near the horizons at  $r_{\pm}$ , the topology reduces to  $\mathcal{C}^2 \times \mathcal{S}^2$  if we set

$$k_{\pm} = \frac{1}{2} \left| h'(r_{\pm}) \right|. \quad (7)$$

$r_{\pm}$  are the solutions of  $h(r_{\pm}) = 0$  and are related to  $m$  and  $\lambda$  through

$$r_+ = \sqrt{\frac{4}{\lambda}} \cos\left(\frac{\xi + 4\pi}{3}\right), \quad r_- = \sqrt{\frac{4}{\lambda}} \cos\left(\frac{\xi}{3}\right), \quad \cos(\xi) = -3m\sqrt{\lambda}, \quad (8)$$

with  $\xi$  in the range  $(\pi, \frac{3\pi}{2}]$  and,  $0 \leq 9m^2\lambda < 1$  [1]. Note that  $k_{\pm}$  are the values of the surface gravity<sup>1</sup> on the horizons, and the Gibbons-Hawking temperatures  $T_{\pm}$  associated with the respective horizons are

$$T_{\pm} = \frac{k_{\pm}}{2\pi}. \quad (9)$$

It is thus clear for the manifold defined this way that there are conical defects if  $T$  is not equal to  $T_{\pm}$ .

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<sup>1</sup> It may be more natural to include a normalization factor in defining the surface gravity(see for instance Ref.[8]). We thank R. Bousso for drawing our attention to this. However, according to Eqs.(A6)-(A9) of Ref.[8] the factor cancels in required periods and therefore none of our conclusions will be affected.

Following Ref.[5], the contributions of the conical singularities to the action are

$$-\frac{1}{16\pi} \int_{\Sigma} d^4x \sqrt{g} R = -\frac{1}{4}(1 - \frac{T_-}{T})A_- + \frac{1}{4}(1 - \frac{T_+}{T})A_+. \quad (10)$$

The relative sign difference in the above expression reflects the opposite orientations of the normals at the horizons with respect to  $dr$ . The conical contributions vanish when the area of the corresponding horizons  $A_{\pm}$  coincide and  $T_- = T_+$ . The conical contributions are obviously nontrivial otherwise.

We also need the contribution from the non-singular set  $M/\Sigma$  to compute the total contribution to  $I_g$  in Eq.(1). By integrating  $r$  from  $r_-$  to  $r_+$  and  $\tau$  from 0 to  $\beta$ , the contribution to the Einstein-Hilbert action from the non-singular set is

$$\begin{aligned} -\frac{1}{16\pi} \int_{M/\Sigma} d^4x \sqrt{g}(R - 2\lambda) &= -\frac{\lambda}{8\pi} \int_{M/\Sigma} d^4x \sqrt{g} \\ &= -\frac{\lambda\beta}{6}(r_+^3 - r_-^3). \end{aligned} \quad (11)$$

Summing the contributions of Eqs.(10) and (11), we have the tree level action (with  $I_m \equiv 0$ ) with temperature dependence as

$$I_g = -\beta(\lambda r_+^3/3 - m) - \pi(r_-^2 - r_+^2). \quad (12)$$

We may consider extremizing and doing thermodynamics with this action, but it does not seem to yield sensible results in the off-shell approach. For instance, the entropy in this approach with conical singularities is

$$S_{Con.sing.} = \beta \frac{\partial I_g}{\partial \beta} - I_g = \pi(r_-^2 - r_+^2). \quad (13)$$

At this point, it is appropriate to spell out a few subtleties and difficulties associated with the formulation with conical singularities. First of all, there is a subtlety with regard to the correct sign of the action. When the entropy is evaluated using this approach for the pure de Sitter configuration<sup>2</sup>, the result is *negative*. Explicitly,

$$S_{Con.sing.} = \beta \frac{\partial I_g}{\partial \beta} - I_g = -\pi r_+^2. \quad (14)$$

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<sup>2</sup>The  $S^4$  de Sitter configuration is interesting from the thermodynamic viewpoint in a number of ways. In quantum gravity, it may be necessary to have a nonvanishing cosmological constant. The Gibbons-Hawking temperature of the de Sitter solution, which is the configuration with the greatest symmetry and a possible ground state in quantum cosmology, is proportional to  $\sqrt{\lambda}$ , and the exact vanishing  $\lambda$  limit may be a physically unattainable zero temperature limit in quantum gravity [9]. The de Sitter solution also appears to violate Nernst's theorem explicitly since its entropy which is proportional to the area of the horizon and inversely proportional to  $\lambda$  does not go to zero with vanishing temperature.

The absolute value is the correct Gibbons-Hawking entropy for pure de Sitter manifold with cosmological horizon at  $r_+$ . This is in contradistinction with the pure Schwarzschild case where the action as in Eq.(1) gives the correct sign and magnitude of the Bekenstein-Hawking entropy and also the correct positive energy equal to the Arnowitt-Deser-Misner mass  $m$ [5]. We emphasize that *on-shell* calculations for the de Sitter solution with  $\beta = 2\pi r_+$  gives the correct positive result for the entropy because  $I_g$  in Eq. (1) *without conical singularities* leads to  $I_g = -S = -\pi r_+^2$ [10]. We may try to choose the action to be the negative of that in Eq.(1) but that convention will lead to problems with the pure Schwarzschild case. The entropy calculated from the method with conical singularities therefore may or may not coincide with the on-shell value even in situations where there is but a single horizon. Moreover, it can even lead to non-positive values of  $S_{Con.Sing.}$ .

Secondly, there are difficulties associated with the formulation with conical singularities if we were to apply it to QFT of matter fields in curved spacetimes with more than one horizon. Physically, it is important to include matter but from Eq.(2), on integrating out the quantum field  $\phi$  in a fixed background metric, the effective action is naively expected to be [11]

$$\begin{aligned} I_{eff}[g_{\mu\nu}, \beta, \lambda] &= \int_M d^4x \sqrt{g} \left\{ \frac{-1}{16\pi G_{ren}} (R - 2\lambda_{ren}) \right. \\ &\quad \left. + c_1 R^2 + c_2 R^{\mu\nu} R_{\mu\nu} + c_3 R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} \right\} + \text{finite terms.} \quad (15) \end{aligned}$$

This is for smooth manifolds. Curvature-squared quantum corrections also contribute to the conformal anomaly which is related to the Hawking radiation, and are thus physically relevant to the thermal feature of spacetime [12]. However, when conical singularities are present, they result in Dirac  $\delta$  singularities in the curvatures[5]. Components of the curvature tensor have to be defined as distributions and integral characteristics of quadratic and higher powers of the curvature do not have strict meaning. We may assume all the higher-order renormalized coefficients of curvature-squared terms vanish identically to bypass this difficulty i.e. to assume that the only effect of quantized matter is just to renormalize the gravitational constant  $G$  (which has been set to unity in our convention) and the cosmological constant  $\lambda$ . However this is due more to expediency than to compelling physical arguments. As was pointed out in Ref.[4], the trace of the heat kernel operator turns out to be well-defined, and we may compute the QFT contributions from the asymptotic expansion of the trace of the heat kernel operator

$$Tr(K) = Tr(\exp(-s\Delta)) = \frac{1}{4\pi s^2} (a_0 + s a_1 + s^2 a_2 + \dots). \quad (16)$$

However it is still necessary to *assume* potential terms in the Laplacian operator are defined only on  $M/\Sigma$  and do not include singular terms. In general, the conical contributions to the coefficients  $a_i$  do not vanish. Moreover, for spin 3/2 and spin 2 fields, even when the on-shell value of  $\beta$  is taken afterwards, the trace of the heat

kernel differs from the trace on smooth manifolds. In the case of the pure black hole where there is but a single horizon, “renormalizations”<sup>3</sup> can be done and the contributions of curvature-squared terms at equilibrium temperature (at which the conical defect disappears) can be taken into account [5]. The crucial difference is that this cannot be done for the relevant multihorizon scenario here because it is impossible to *simultaneously* eliminate conical defects at both horizons.

While these hurdles do not conclusively show that there is no sensible way to define QFT in S-dS spacetime via the conical method, it is nevertheless true that that different ways of accounting for the horizons lead to different results. There is thus ambiguity as to which of the methods is “appropriate”. Some of the methods are covered in a review of on-shell vs. off-shell computations[13]. The discussion includes the “brick-wall” method with Dirichlet boundary conditions and a cut-off distance from the horizon, the “blunt cone” method where the conical singularities are smoothed away by a deformation parameter, the “volume cut-off” formalism, the method with conical singularities, and on-shell computations for the pure black hole.

Actually, the very meaning of “on-shell” for the case of Schwarzschild-de Sitter with two horizons is problematic from the perspective of the conical method, and is so far undefined. But this will be pursued in the next section.

### III. Kruskal extension of the S-dS spacetime.

We emphasize that different ways of accounting for the horizons lead to different results for the effective action. We propose that a more natural way to account for the horizons is to consider the Kruskal extension of the manifold and then define QFT on the Euclidean section. We draw a lesson first from Kruskal extension of the pure Schwarzschild solution and see why it leads to  $\beta = 8\pi m$  naturally.

(a) The pure black hole metric is

$$ds^2 = -h(r)dt^2 + h^{-1}(r)dr^2 + r^2d\Omega^2 \quad (17)$$

with  $h(r) = \frac{(r-2m)}{r}$ . By defining  $u = t - r^*$  and  $v = t + r^*$ , with

$$\begin{aligned} r^* &= \int \frac{r}{(r-2m)}dr \\ &= r + 2m \ln(r-2m), \end{aligned} \quad (18)$$

the metric can be transformed to

$$ds^2 = -h(r)dudv + r^2d\Omega^2. \quad (19)$$

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<sup>3</sup> In the case of quantized matter in Schwarzschild *background* of fixed mass, the black hole mass is treated as a macroscopic parameter so that  $Z(\beta, G, m)$ .

The Kruskal extension can be done with coordinates

$$u' = -e^{-ku}, \quad v' = e^{kv} \quad (20)$$

where  $k = \frac{1}{2}dh/dr|_{r=2m} = \frac{1}{4m}$  is the surface gravity at the horizon. In terms of Kruskal coordinates, the metric becomes

$$\begin{aligned} ds^2 &= -h(r) \frac{du}{du'} \frac{dv}{dv'} du' dv' + r^2 d\Omega^2 \\ &= -\frac{1}{k^2 r} e^{-2kr} du' dv' + r^2 d\Omega^2. \end{aligned} \quad (21)$$

By rewriting

$$t' = (u' + v')/2 \quad r' = (v' - u')/2, \quad (22)$$

we have

$$ds^2 = \frac{1}{k^2 r} e^{-2kr} (dr'^2 - dt'^2) + r^2 d\Omega^2 \quad (23)$$

and  $r'^2 - t'^2 = -u'v' = e^{2kr}(r - 2m)$ . So the Euclidean section for which the metric is positive-definite can be defined by a Wick rotation of  $t$  which makes  $t'$  pure imaginary; and hence for  $r \geq 2m$  only. Moreover,  $\tau = it$  has period  $2\pi/k$  since

$$t' = (u' + v')/2 = e^{kr^*} \sinh(-ik\tau), \quad (24)$$

and  $(u', v')$  defines  $\tau$  up to multiples of  $2\pi/k = 8\pi m$ . This periodicity for  $\tau$  is precisely the condition for the manifold to be free of conical singularity, but *it emerges naturally as a consistency condition of the Euclidean section of the Kruskal extension*. In the Kruskal extension, the form of the metric is non-singular at the horizon. Euclidean QFT can be constructed for the Kruskal extension with the on-shell restriction of  $\beta = 8\pi m$ .

Note that if we neglect the spherically symmetric  $d\Omega^2$ -part, only one coordinate patch is required for the Kruskal extension because the topology is  $R^2$  and the topology of the four-manifold is  $R^2 \times S^2$ . The Euclidean section has Euler number  $\chi = 2$ . For the Schwarzschild-de Sitter metric, the Kruskal extension needs more than one coordinate patch even if we neglect the spherically symmetric part of the metric because there are now two horizons with unequal surface gravity. However it is clear that the Euclidean section of the Kruskal extension of the black hole *includes the horizon* and also yields *consistency conditions* on the periodicity of  $\tau$ . We shall therefore use the Kruskal extension to consistently incorporate both horizons of the Schwarzschild-de Sitter manifold in the Euclideanization, and also to deduce the consistency conditions that are required. In this manner, boundary conditions for quantized matter fields in the Schwarzschild-de Sitter background will emerge naturally from the Euclidean quantum field theory. Moreover, as we shall see, the conditions that arise are rather interesting.

(b) The Schwarzschild-de Sitter metric has the form

$$ds^2 = -h(r)dt^2 + h^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (25)$$

with

$$h(r) = \frac{\lambda}{3r}(r_+ - r)(r - r_-)(r + r_+ + r_-). \quad (26)$$

It is a solution of Einstein's equations with cosmological constant  $\lambda$  if

$$3/\lambda = r_+^2 + r_+r_- + r_-^2, \quad 6m/\lambda = r_+r_-(r_+ + r_-). \quad (27)$$

The metric is a priori defined for the region between the horizons but there can be a Kruskal extension.<sup>4</sup>

The surface gravity at the horizons are given by

$$k_{\pm} = \frac{1}{2}|dh(r)/dr|_{r=r_{\pm}}, \quad (28)$$

or

$$k_{\pm} = \frac{\lambda}{6r_{\pm}}(r_+ - r_-)(2r_{\pm} + r_{\mp}). \quad (29)$$

The horizons have different values of surface gravity and their ratio satisfy  $0 < k_+/k_- \equiv \alpha \leq 1$ .

Similarly, we define

$$u \equiv t - r^* \quad v \equiv t + r^* \quad (30)$$

with

$$\begin{aligned} r^* &= \int h^{-1}(r)dr \\ &= \frac{1}{2k_-} \ln(r - r_-) - \frac{1}{2k_+} \ln(r_+ - r) + \left(\frac{1}{2k_+} - \frac{1}{2k_-}\right) \ln(r + r_+ + r_-). \end{aligned} \quad (31)$$

In terms of these coordinates,

$$ds^2 = -h(r(u, v))dudv + r^2(u, v)(d\theta^2 + \sin^2\theta d\phi^2), \quad (32)$$

with  $r$  defined implicitly by  $r^*(r) = (v - u)/2$ .

We may cover the Kruskal extension by two coordinates patches  $(u_{\pm}, v_{\pm})$ .  $(u_+, v_+)$  is valid for  $r > r_-$ , which includes the outer or cosmological horizon but not the inner or blackhole horizon; while  $(u_-, v_-)$  is valid for  $r < r_+$ , and includes the inner but not the outer horizon. These coordinates are

$$u_{\pm} = \pm e^{\pm k_{\pm} u}, \quad v_{\pm} = \mp e^{\mp k_{\pm} v}. \quad (33)$$

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<sup>4</sup>See, for instance, Ref.[14].

Thus

$$\begin{aligned} du_{\pm} dv_{\pm} &= k_{\pm}^2 e^{\pm k_{\pm}(u-v)} du dv \\ &= k_{\pm}^2 e^{\mp 2k_{\pm}r^*} (dt^2 - dr^{*2}), \end{aligned} \quad (34)$$

and

$$\begin{aligned} ds^2 &= -h(r) \frac{du}{du_{\pm}} \frac{dv}{dv_{\pm}} du_{\pm} dv_{\pm} + r^2 d\Omega^2 \\ &= -h_{\pm} du_{\pm} dv_{\pm} + r^2 d\Omega^2, \end{aligned} \quad (35)$$

where

$$h_{-} = \frac{\lambda}{3k_{-}^2 r} (r_{+} - r)^{(1+\alpha^{-1})} (r + r_{+} + r_{-})^{(2-\alpha^{-1})}, \quad (36)$$

and

$$h_{+} = \frac{\lambda}{3k_{+}^2 r} (r - r_{-})^{(1+\alpha)} (r + r_{+} + r_{-})^{(2-\alpha)}. \quad (37)$$

$h_{-}$  is valid for the patch with  $0 < r < r_{+}$  and  $h_{+}$  for that with  $r > r_{-}$ . Therefore it is clear that the metric is nonsingular (except at  $r = 0$ ) and also nonvanishing in each of the respective coordinate patch. The overlap of the patches occurs in the region  $r_{-} < r < r_{+}$  where the coordinates are related by

$$u_{+} = -e^{(k_{+}+k_{-})u} u_{-}, \quad v_{+} = -e^{-(k_{+}+k_{-})v} v_{-}. \quad (38)$$

We may also note that by a Wick rotation of  $t$ , the metric becomes positive definite for  $r_{-} \leq r \leq r_{+}$  since with  $\tau = it$  the metric is

$$ds^2 = h_{\pm} k_{\pm}^2 e^{\mp 2k_{\pm}r^*} (d\tau^2 + dr^{*2}) + r^2 d\Omega^2. \quad (39)$$

In terms of Kruskal coordinates  $(u_{\pm}, v_{\pm})$ , the metric is

$$ds^2 = h_{\pm} |du_{\pm}|^2 + r^2 d\Omega^2. \quad (40)$$

since  $u_{\pm}$  become complex conjugates of  $-v_{\pm}$  after Euclideanization. To satisfy  $|u_{\pm}|^2 \geq 0$ , the Euclidean section is defined only for  $r_{-} \leq r \leq r_{+}$ ; and it can be shown that the horizons at  $r_{\pm}$  correspond to the origins  $u_{\pm} = 0$ . Expression (40) shows that the extended Kruskal Riemannian manifold exhibits no singular behaviour at the horizons. In the overlap region with  $r_{-} < r < r_{+}$ ,

$$\begin{bmatrix} u_{+} \\ v_{+} \end{bmatrix} = \begin{bmatrix} -e^{(k_{+}+k_{-})(-i\tau-r^*)} & 0 \\ 0 & -e^{-(k_{+}+k_{-})(-i\tau+r^*)} \end{bmatrix} \begin{bmatrix} u_{-} \\ v_{-} \end{bmatrix}. \quad (41)$$

So the transition function is single-valued only if  $\beta$ , the period of  $\tau$ , is an integer multiple of  $\frac{2\pi}{(k_{+}+k_{-})}$ . However, there are stronger consistency conditions. Since

$$(u_{\pm} + v_{\pm})/2 = e^{\mp k_{\pm}r^*} \sinh(-ik_{\pm}\tau), \quad (42)$$

this means that  $(u_{\pm}, v_{\pm})$  only define values of  $\tau$  up to integer multiples of  $2\pi/k_+$  and  $2\pi/k_-$  in each patch. But  $(u_{\pm}, v_{\pm})$  are also well-defined coordinates in the overlap. Therefore the translation  $\tau \rightarrow \tau + \beta$  which leaves *both* sets of coordinates  $(u_{\pm}, v_{\pm})$  invariant must be such that  $\beta$  has to be an integer multiple of *both*  $2\pi/k_+$  and  $2\pi/k_-$ . This means that in fact  $\beta$ , the period of  $\tau$ , is therefore the *lowest common multiple* of  $\frac{2\pi}{k_+}$  and  $\frac{2\pi}{k_-}$ . It is easy to check that this is sufficient (although not necessary) for the transition function to be single-valued. The latter is a weaker condition. There is however an interesting relation: Let  $n_{\pm}$  be relatively prime positive integers such that  $\beta \equiv 2\pi n_{\pm}/k_{\pm}$ . Thus  $0 < \alpha = k_+/k_- = n_+/n_- \leq 1$  is rational.<sup>5</sup> Then  $(n_+ + n_-) = \frac{\beta}{2\pi}(k_+ + k_-)$  or

$$\beta = 2\pi \frac{(n_+ + n_-)}{(k_+ + k_-)}. \quad (43)$$

By comparing with Eqs.(38) and (40), we see that under  $\tau \rightarrow \tau + \beta$ , the transition functions of the  $u_{\pm}$  and  $v_{\pm}$  coordinates gets multiplied by  $\exp[\mp i2\pi(n_+ + n_-)]$ ; and  $(n_+ + n_-)$  is the *winding number* of the transition function.

## IV. Topological considerations.

The Lorentzian Kruskal extension of S-dS is known to exhibit a multi-sheeted structure with the Penrose diagram showing repeating units [10]. Thus there is the question of what one means by the Euclidean section with  $r_- \leq r \leq r_+$ , and what the actual topology (specifically the Euler number) is. The consistency condition that we uncovered in the previous section is related to these issues.

We first compute the Euler number of the Euclidean manifold *with conical singularities* [5]. This is given by

$$\chi[M] = \chi[M/\Sigma] + \chi[\Sigma], \quad (44)$$

with the regular contribution

$$\chi[M/\Sigma] = \frac{1}{32\pi^2} \int_{M/\Sigma} d^4x \sqrt{g} (R^2 - 4R_{\mu\nu}^2 + R_{\mu\nu\alpha\beta}^2), \quad (45)$$

and the contributions from the conical singularities,

$$\chi[\Sigma] = \sum_{\pm} \left(1 - \frac{\beta}{\beta_{\pm}}\right) \chi_2[\Sigma_{\pm}]. \quad (46)$$

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<sup>5</sup>The metric of Eqs.(35)-(37) is still well-defined for irrational exponents through  $a^x = \exp(x \ln a)$ ,  $a > 0$ . Restricting  $\alpha = k_+/k_-$  to rational numbers yields finite  $\beta$ . Rational numbers are also dense in the system of real numbers. If  $\alpha$  is irrational,  $\beta$  becomes larger and larger with improving approximations of irrationals by rationals.

$\beta_{\pm} = 1/T_{\pm} = 2\pi/k_{\pm}$  are the periods associated with the horizons. For the Euclidean S-dS manifold, the explicit computation is straightforward. The results are

$$\chi[M/\Sigma] = 2\beta\left(\frac{1}{\beta_+} + \frac{1}{\beta_-}\right), \quad (47)$$

$$\begin{aligned} \chi[\Sigma] &= \chi_2[S^2]\left[2 - \beta\left(\frac{1}{\beta_+} + \frac{1}{\beta_-}\right)\right] \\ &= 4 - 2\beta\left(\frac{1}{\beta_+} + \frac{1}{\beta_-}\right) \end{aligned} \quad (48)$$

since  $\chi_2[S^2] = 2$ . Thus, as expected,  $\chi[M]$  for S-dS is *always* equal to 4 and is *independent of  $\beta$*  if the horizons are incorporated as conical singularities<sup>6</sup>. In this sense, allowing for conical singularities preserves the topological information which remains constant under deformations of the parameters  $m, \lambda$  and  $\beta$ ; and conical singularities seem rather appealing as seeds for condensation and remnants of black hole evaporation. For instance, the specific relation between  $\beta$  and the value of  $m$  which extremizes (at fixed  $\lambda$ ) the action of Eq.(12) can be worked out. There are actually critical values of  $\beta$  for which  $m$  approaches  $0^+$  and becomes larger as  $\beta$  is varied. However, when we wish to consider higher order curvature terms, there are ambiguities and difficulties associated with QFT contributions if horizons are accounted for as conical singularities.

In contrast, the Euclidean section of the Kruskal extension gives a different result for the Euler number. Recall that  $\beta$  has to be the lowest common multiple of  $2\pi/k_+$  and  $2\pi/k_-$  to satisfy the consistency condition discussed in the previous section so that the translation  $\tau \rightarrow \tau + \beta$  is a symmetry. In the Kruskal extension there are no conical singularities, and Eq.(44) yields

$$\chi[M] = 2\beta\left(\frac{1}{\beta_+} + \frac{1}{\beta_-}\right) = 2(n_+ + n_-). \quad (49)$$

In the last step above, we have substituted the values of  $\beta$  from Eq.(43) and  $\beta_{\pm} = 2\pi/k_{\pm}$ . The Euler number is therefore integer and even. The former is consistent with the Euler number of Riemannian manifolds, and the latter is due to the spherical symmetry as  $\chi_2[S^2] = 2$ . As mentioned previously,  $(n_+ + n_-)$  is the winding number of the transition function displayed in Eq.(41). Therefore we may also write Eq.(43) as

$$\beta = \frac{\pi\chi}{(k_+ + k_-)}. \quad (50)$$

However the Euler number is not fixed to be exactly 4. The reason is that basic repeating Euclidean unit for which  $\tau \rightarrow \tau + \beta$  is a symmetry depends on both  $k_+$  and  $k_-$ .

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<sup>6</sup>For the pure Schwarzschild black hole, similar computations yield  $\chi = 2$ .

In the approach with conical singularities, the Euler number is divided between the singular and regular parts of the manifold. These two values can be adjusted by changing  $\beta$  although their sum is always 4. Computations of other invariants further differentiate between the alternatives. With conical singularities, the four-volume is  $\frac{4\pi\beta}{3}(r_+^3 - r_-^3)$  and is a function of  $\beta$  which is independent, while  $\beta$  is not arbitrary for the case with Kruskal extension. In the method with conical singularities, conical contributions to the action given by Eq.(10) also do not vanish in general. Thus even if some invariants can be matched by certain choices of  $\beta$ , others will not be. Only the limiting case of  $k_+ = k_-$  allows for a correspondence between the formalism with conical singularities and the results from using the Kruskal extension, since in this limiting case conical defects on both horizons can be eliminated by a single choice of  $\beta$  in the formalism with conical singularities.

## V. Remarks

We have discussed some of the difficulties with QFT contributions in the off-shell approach if the horizons are to be accounted for as conical singularities of the Euclidean section. The problems becomes more transparent and acute in scenarios with more than one horizon; and we have considered the explicit example of Schwarzschild-de Sitter. A more natural way to incorporate the horizons emerges from considering the Kruskal extension and then constructing the QFT on the Euclidean section. In this manner no conical singularites are introduced but the horizons with their unequal surface gravity lead to natural selection rules or consistency conditions on the periodicity of the Euclidean time variable; and suggests that these are the natural boundary conditions that should be imposed upon such a QFT and the quantum states. Moreover, this implies that thermal states with  $\beta$  being lowest common multiple of  $2\pi/k_+$  and  $2\pi/k_-$  exist. In this approach,  $\beta$  can no longer assume arbitrary off-shell values but is completely determined by the stated consistency condition. Since there are no conical singularities, the one-loop effective action on integrating out quantized matter fields will contain the usual terms (and counterterms) without arbitrary  $\beta$ -dependent contributions and Dirac delta singularities in terms quadratic in the curvatures. Although we have not set up an explicit quantum field theory and completed the calculation of the stress tensor in 4-d, there is support for our conjecture. The existence of quantum states for S-dS whose stress tensor is static has been shown explicitly for the 2-d case of S-dS for which the angular dependence is neglected [14]. Our results therefore also offers an understanding of this from the Euclidean approach. For the pure black hole and de Sitter configurations, our proposal is equivalent to the on-shell requirements but it is interesting to note that the proposal also serves to give meaning to the concept of “on-shell” for the Schwarzschild-de Sitter manifold; and may be generalizable to even more complicated scenarios.

The Schwarzschild-de Sitter example also illustrates the much richer interplay

among horizons, QFT and topology that can occur when the cosmological constant is *not neglected* in black hole scenarios. It will be interesting to investigate the stability when back reactions are taken into account. For instance since  $\beta$  is given by the lowest common multiple condition, it can vary wildly with deformations of  $k_{\pm}$  if there are no further restrictions. However, it is important to note that conservation of topological Euler number implies that  $(n_+ + n_-)$  should be constant. Thus within each topological sector where this number is conserved,  $\beta$  varies inversely with  $(k_- + k_+)$  (see Eq. (50)). More chaotic behaviour can of course happen in quantum gravity when tunneling between different sectors and also violations of toplogical conservation laws are allowed.

Finally, on possible interesting oscillating behaviour for evaporating black holes with cosmological constant, we feel that it is important to distinguish between the eternal and the evaporating case. The S-dS solution that we have is already a *four*-manifold. Its mass parameter  $m$  does not increase or decrease with the time variable  $t$ , barring for instance superspace descriptions in quantum gravity where some other degree of freedom is chosen as “time”. An evaporating or anti-evaporating black hole with cosmological constant for which the size of the inner horizon increases or decreases with time is a different four-manifold from the eternal case we have considered. Therefore the requirements on the periodicity (if there are any obvious ones following our prescription) are quite clearly different since the Kruskal extension will be that of another four-manifold. Thus our arguments do not necessarily imply that an evaporating black hole has arbitrarily large jumps in its Euclidean period. As for the superspace or quantum gravity context, we are neither able to prove nor disprove possible large jumps in the period.

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